FINAL DIFFERENTIAL EQUATIONS SUMMARY

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1. Linear Algebra

1.1. Vector Spaces .

Definition 1. A *vector space* over $\mathbb{R}$ is a set V with operations

$$+ : V \times V \to V$$
$$\cdot : \mathbb{R} \times V \to V$$

satisfying for all $x, y, z \in V$, $a, b \in \mathbb{R}$

- $x + y = y + x$
- $x + (y + z) = (x + y) + z$
- There exists a unique element $0 \in V$ with that property that $0 + x = x$.
- there is a unique element $-x \in V$ such that $x + (-x) = 0$.
- $1 \cdot x = x$
- $(ab) \cdot x = a \cdot (b \cdot x)$
- $a \cdot (x + y) = a \cdot x + a \cdot y$
- $(a + b) \cdot x = a \cdot x + b \cdot x$

Definition 2. A *linear subspace* of a vector space V, is a subset $W \subset V$ which is closed under addition and multiplication, i.e. for all $w_1, w_2 \in W$ and $a \in \mathbb{R}$, $w_1 + w_2 \in W$ and $aw_1 \in W$. Equivalently, it is a subset $W \subset V$ which is itself a vector space under the same operations $+, \cdot$ as V.

Definition 3. A set of vectors $\{e_1, e_2 \dots, e_n\} \subset V$ is called *linearly independent* if the linear equation

$$c_1e_1 + c_2e_2 + \dots c_ne_n = 0$$

where $c_1, \dots, c_n \in \mathbb{R}$ has a unique solution $c_1 = c_2 = \dots = c_n = 0$. Equivalently, a set $\{e_1, e_2 \dots, e_n\}$ is linearly independent if none of the vectors e_i is a linear combination of the others.

Definition 4. *Linear span* of a set of vector $\{e_1, \dots, e_n\} \subset V$ is the set of vector w of the form

$$w = c_1e_1 + \dots + c_ne_e$$

with $c_1, \dots, c_n \in \mathbb{R}$. In words, it's the set of vectors expressible as a linear combination of vectors $e_1, \dots, e_n$.

Definition 5. The *dimension* of a vector space V is the least number n such that there are n vectors $\{e_1, \dots, e_n\} \subset V$ which span V.

Theorem 6. *Let V be a vector space. The following are equivalent*

- *The dimension of V is n.*
- *The maximal number of linearly independent vectors in V is n.*
- *There exists a linearly independent set of vectors $\{e_1, \dots, e_n\} \subset V$* which spans V.
- *Any set $\{e_1, \dots, e_n\} \subset V$ of linearly independent vectors spans V.*

Definition 7. Let V be a vector space of dimension n. A set $\{e_1, \dots, e_n\}$ of linearly independent vectors is called a *basis* of V.

Theorem 8. *Let V be a vector space and $\{e_1, \dots, e_n\}$ be a basis of V. Then for any vector $v \in V$ there are unique $c_1, \dots, c_n \in \mathbb{R}$ such that*

$$v = c_1e_1 + \dots + c_ne_n.$$

Example 9. The most important example to understand is the vector space

$$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} |x_i \in \mathbb{R} \right\}$$

which is denoted $\mathbb{R}^n$. The set $\{e_1, \dots, e_n\}$ where

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \longleftarrow \text{i'th row}$$

is a basis of $\mathbb{R}^n$. Indeed, $\{e_1, \dots, e_n\}$ is linearly independent since

$$c_1 e_1 + \cdots + c_n e_n = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = 0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

if and only if $c_1 = \cdots = c_n = 0$. Also, $\{e_1, \dots, e_n\}$ spans $\mathbb{R}^n$ since for any $v = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, we have

$$v = x_1 e_1 + \cdots + x_n e_n.$$

In particular, the dimension of $\mathbb{R}^n$ is n. A set of vectors $\{f_1, \dots, f_m\} \subset \mathbb{R}^n$ where $f_i = \begin{pmatrix} f_{1i} \\ f_{2i} \\ \vdots \\ f_{ni} \end{pmatrix}$ is linearly independent if the equation

$$c_1 f_1 + \dots c_m f_m = 0$$

has a unique solution for $c_1, \dots, c_m$. The left hand side is

$$\begin{pmatrix} c_1 f_{11} + c_2 f_{12} \cdots + c_m f_{1m} \\ c_1 f_{21} + c_2 f_{22} + \cdots + c_m f_{2m} \\ \vdots \\ c_1 f_{n1} + c_2 f_{n2} + \dots c_m f_{nm} \end{pmatrix} = \begin{pmatrix} f_{11} & f_{12} & \dots & f_{1m} \\ f_{21} & f_{22} & \dots & f_{2m} \\ \vdots & \ddots & \ddots & \vdots \\ f_{n1} & f_{n2} & \dots & f_{nm} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}.$$

If we denote the matrix $\begin{pmatrix} f_{11} & f_{12} & \dots & f_{1m} \\ f_{21} & f_{22} & \dots & f_{2m} \\ \vdots & \ddots & \ddots & \vdots \\ f_{n1} & f_{n2} & \dots & f_{nm} \end{pmatrix}$ by F, then the set $\{f_1, \dots, f_m\}$ is linearly independent if and only if the system of equations

$$F \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} = 0$$

has a unique solution.

For example the set of vectors $\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\} \subset \mathbb{R}^2$ are linearly independent since if

$$c_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 0$$

then

$$\begin{aligned} 2c_1 + c_2 &= 0 \\ 3c_1 + 3c_2 &= 0. \end{aligned}$$

From the second equation we get $c_1 = -c_2$ and plugging it into first equation we get $-2c_2 + c_2 = 0$ which implies $c_2 = c_1 = 0$. On the other hand, the set $\left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \end{pmatrix} \right\}$ is linearly dependent since

$$2\begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 6 \end{pmatrix} = 0.$$

The subset

$$\{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} | 3x_1 + 2x_2 = 0\}$$

is a linear subspace of $\mathbb{R}^n$ while the subspaces

$$\{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} | 3x_1 + 2x_2 = 3\},$$

$$\{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} | 3x_1^2 + 2x_2 = 0\}$$

are not linear subspaces.

1.2. **Matrices.** An $n \times m$ matrix is a collection of nm numbers arranged into n rows and m columns. Given a matrix A, we denote the entry in the i'th row and j'th column by A_{ij}.

Definition 10. Given an $n \times m$ matrix

$$A = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix}$$

and an $n' \times m'$ matrix

$$B = \begin{pmatrix} b_{11} & \dots & b_{1m'} \\ \vdots & \ddots & \vdots \\ b_{n'1} & \dots & b_{n'm'} \end{pmatrix},$$

the product AB is defined only if $m = n'$ in which case AB is a $n \times m'$ matrix with entries

$$\{AB\}_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}.$$

Pictorially, $\{AB\}_{ij}$ is the scalar product of i'th row of A and j'th column of B.

$$\begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{im} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix} \begin{pmatrix} b_{11} & \dots & b_{1j} & \dots & b_{1m'} \\ \vdots & \ddots & \dots & \ddots & \vdots \\ b_{n'1} & \dots & b_{n'j} & \dots & b_{n'm'} \end{pmatrix}$$

Of particular importance is the case where B is a $m \times 1$ matrix, or in other words a column vector of height m, or an element of $\mathbb{R}^m$. In that case AB is a column vector of height n. Pictorially In particular,

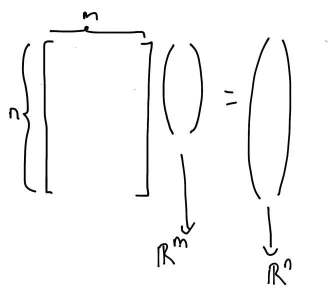

FIGURE 1.1.

we can think of a $n \times m$ matrix as a transformation that sends elements of $\mathbb{R}^m$ to elements of $\mathbb{R}^n$.

Matrix multiplication can be used to concisely express a system of linear equations. In particular,

$$\begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

is a short hand for the following system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1m}x_m &= b_1 \\ \vdots &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nm}x_m &= b_{n.} \end{aligned}$$

1.3. **Determinants.**

Definition 11. The determinant of a $n \times n$ matrix A is defined by

$$\det A = \sum_{j_1,\dots,j_n} \epsilon_{j_1\dots j_n} A_{1j_1} A_{2j_2} \dots A_{nj_n}$$

where the sum is over permutation, i.e. $j_k \neq j_l$ if $k \neq l$, and $\epsilon_{j_1\dots j_n}$ is the sign of the permutation.

For $n = 2, 3$ we have

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - afh - bdi - ceg.$$

For $n \geq 4$, the exact formula becomes cumbersome. There are two methods for simplifying the computation of determinant: row reduction and expansion with respect to a row or a column.

Row/Column expansion: Denote by $A(i|j)$ the matrix one obtains from A by removing i'th row and j'th column. Then for a fixed k we have

$$\det A = \sum_{l=1}^{n} (-1)^{l+k} A_{lk} \det A(l|k) = \sum_{l=1}^{n} (-1)^{l+k} A_{kl} \det A(k|l)$$

where the expression $\sum_{l=1}^{n} (-1)^{l+k} A_{lk} \det A(l|k)$ corresponds to expansion with respect to k'th column and the expression $\sum_{l=1}^{n} (-1)^{l+k} A_{kl} \det A(k|l)$ corresponds to expansion with respect to k'th row. As an example, we have

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 3 \\ 5 & 4 & 2 \end{pmatrix} = 1 \det \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix} - 4 \det \begin{pmatrix} 2 & 3 \\ 4 & 2 \end{pmatrix} + 5 \det \begin{pmatrix} 2 & 3 \\ 5 & 3 \end{pmatrix}.$$

Row reduction. We use the following properties of the determinant function

- If you interchange two rows(columns), the value of the determinant changes sign
- If you add a multiple of a row(column) to **another** row(column) then determinant does not change
- If you multiply any row(column) by a constant, the determinant is multiplies by the same constant
- $\det \begin{pmatrix} a_{11} & * & * & * \\ 0 & a_{22} & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & a_{nn} \end{pmatrix} = a_{11}a_{22}\dots a_{nn}$

As an example we have

$$\det \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 4 \\ 3 & 2 & 2 \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & -4 \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{pmatrix} = -4$$

1.4. Matrix Inverse.

Definition 12. The inverse of a $n \times n$ matrix A is another matrix, which we denote A^{-1} such that

$$A^{-1}A = AA^{-1} = I$$

where

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is called the identity matrix. (Note that $AI = IA = A$ for all $n \times n$ matrices A)

Not all matrices are invertible, but if A is invertible, then the inverse A^{-1} is unique.

Theorem 13. *A matrix A is invertible if and only if* $\det A \neq 0$.

There is an explicit formula for the inverse of a matrix. Let adj(A) be the matrix defined by

$$\operatorname{adj}(A)_{ij} = (-1)^{i+j} \det A(j|i)$$

where $A(j|i)$ is defined in subsection 1.3. Then

$$A^{-1} = \frac{1}{\det A}\operatorname{adj}(A).$$

As one might notice, using this formula requires computing a lot of determinants.

Another way to find an inverse of a matrix is to find a sequence of row operations that transforms the matrix A into the identity matrix and then apply the same row operations to the identity matrix. This works since "applying a sequence of row operations" corresponds to multiplication of A by some matrix B on the left. If the row operations transform A into I, then $BA = I$ (hence $B = A^{-1}$) and applying them to the identity matrix we get $BI = B$. In practice we apply the row operation to A and I at the same time. As an example consider the problem of finding A^{-1} where $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, we then compute

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right) \xrightarrow{R_1 - 2R_2 \to R_1} \left(\begin{array}{cc|cc} 0 & -5 & 1 & -2 \\ 1 & 3 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & -5 & 1 & -2 \end{array}\right) \xrightarrow[R_2 \to -\frac{1}{5}R_2]{R_1 + \frac{3}{5}R_2 \to R_1} \left(\begin{array}{cc|cc} 1 & 0 & \frac{3}{5} & 1 - \frac{6}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{array}\right)$$

and hence $A^{-1} = \begin{pmatrix} \frac{3}{5} & 1 - \frac{6}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix}$.

1.5. **Systems of Linear Equations.** As stated earlier, a system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= b_1 \\ \vdots &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= b_n. \end{aligned}$$

can be compactly written as

$$Ax = b$$

where $A = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$.

The system of equations can have different number of solutions depending on n, m, A, b. Here are few cases you need to understand

(1) If $b = 0$, then there is alway at least one solution, namely $x = 0$ and the space of all solutions is a linear subspace of $\mathbb{R}^m$.
   (a) If $m > n$, then there are non-trivial solutions,
   (b) If $m = n$, then there are non-trivial solutions if and only if $\det A = 0$
(2) If $n = m$, and $\det A \neq 0$ then there is a unique solution given by

$$x = A^{-1}b.$$

In general, to solve this system of equations, one "row reduces" the equation until it is a form that is easy to solve (A is in Echelon form). We will demonstrate this with an example. Consider the following system of equations

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Row reduction gives us

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & 1 \end{array} \right) \xrightarrow{R_2 - 3R_1 \to R_2} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & -2 \end{array} \right)$$

and in particular the original system of equations is equivalent to

$$\begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

which is easy to solve: from the second row we get

$$-2x_2 = -2 \Longrightarrow x_2 = 1$$

and from the first row we get

$$x_1 + 2x_2 = 1 \Longrightarrow x_1 = 1 - 2x_2 = 1 - 2 = -1$$

and the solution is $x = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

As another example, consider

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Row reduction gives us

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 6 & 2 \end{array} \right) \xrightarrow{R_2 - 3R_1 \to R_2} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & -1 \end{array} \right)$$

whose bottom row reads $0 = -1$ and in particular there are no solutions to the original system of equations.

As another example, consider

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Row reduction gives us

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 6 & 2 \end{array}\right) \xrightarrow{R_2-3R_1\to R_2} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

Here, there are no inconsistencies, so a solution exists. Moreover, since one of the rows is exactly zero, the space of solutions has dimension greater than 0. In this case, the columns with no leading terms correspond to free variables which can be set to arbitrary constants. In our example, x_2 is a free variable and we can set it to equal to an arbitrary constant $x_2 = c$ with $c \in \mathbb{R}$. The first row then gives us

$$x_1 + 2x_2 = 1 \Longrightarrow x_1 = 1 - 2x_2 = 1 - 2c$$

and the set of solutions is

$$x = \begin{pmatrix} 1-2c \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

with $c \in \mathbb{R}$. Remark: in the matrix

$$\begin{pmatrix} 1 & 2 & 4 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

columns 2 and 4 correspond to free variables.

1.6. Matrix Diagonalization.

The section is important for the section on linear systems of differential equations.

Definition 14. Given a matrix A, an eigenvector of A is a non-zero column vector v such that

$$Av = \lambda v$$

where $\lambda \in \mathbb{R}$. Equivalently it is a non-zero vector such that

$$(A - \lambda I)v = 0.$$

The number λ is called the corresponding eigenvalue.

In order to find the eigenvectors and corresponding eigenvalues of a matrix A, we first find all λ such that

$$(A - \lambda I)v = 0$$

has a non-trivial solution, or equivalently such that

$$\det(A - \lambda I) = 0.$$

This is called the characteristic polynomial (it has degree n). After we find all such λ, we solve the system of equations

$$(A - \lambda I)v = 0$$

to find all eigenvectors corresponding to different λs.

If λ_0 is a root of $\det(A - \lambda I) = 0$ with multiplicity k, we would like to find k linearly independent eigenvectors corresponding to λ_0. Sometimes there are strictly fewer linearly independent vectors corresponding to the eigenvalue λ_0. In this case we will look for generalized eigenvectors corresponding to λ_0.

Definition 15. A generalized eigenvector of A corresponding to an eigenvalue λ_0 is a vector v satisfying

$$(A - \lambda I)^l v = 0$$

for some l.

Example 16. Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The characteristic polynomial is

$$\det(A - \lambda I) = 0$$

$$(2-\lambda)^2(1-\lambda) = 0$$

whose roots are $\lambda_1 = 1, \lambda_2 = 2$ with multiplicity 2 for eigenvalue λ_2.

To find eigenvectors corresponding to λ_1, we solve

$$(A - \lambda_1)\, v = 0$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} v = 0.$$

The space of solutions is one dimensional with a basis

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

To find eigenvectors corresponding to λ_2, we solve

$$(A - \lambda_2)\, v = 0$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} v = 0.$$

The space of solutions is again one dimensional with a basis

$$v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Since λ_2 had multiplicity 2 and only 1 eigenvector, we look for a generalized vector: v such that

$$(A - \lambda_2)^2\, v = 0$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} v = 0.$$

The space of solutions is 2-dimensional with a basis given by

$$v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2' = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

In particular, the eigenvector v_2 is also a generalized eigenvector of A. The set $\{v_1, v_2, v_2'\}$ is a basis of $\mathbb{R}^3$ consisting of generalized eigenvectors.

2. Systems of Differential Equations

2.1. **Existence/Uniqueness.**

Theorem 17. *Let A be a $n \times n$ matrix, $x^0 = \begin{pmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{pmatrix}$ and $t_0 \in \mathbb{R}$. Then there exists a unique solution $x(t)$ to the initial value problem*

$$\frac{dx}{dt}(t) = Ax(t); \quad x(t_0) = x^0$$

which is defined for all $t \in \mathbb{R}$.

Here, $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ is a $\mathbb{R}^n$ valued function on $\mathbb{R}$ i.e. to each $t \in \mathbb{R}$ it assigns the column vector $\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ in $\mathbb{R}^n$. We will also use notation $\dot{x}$ for $\frac{dx}{dt}$ in the future.

A consequence of the above theorem is that the space of solutions to the system of differential equations

$$\dot{x}(t) = Ax(t) \tag{2.1}$$

is n dimensional. To see this, fix $t_0 \in \mathbb{R}$ and note that

$$x(t) \mapsto x(t_0)$$

is a one-to-one correspondence between the space of solutions and the vector space $\mathbb{R}^n$. Indeed, given an element $x^0 \in \mathbb{R}^n$, the theorem gives the unique solution $x(t)$ to equation 2.1 with the initial value $x(t_0) = x^0$.

Another consequence of the theorem is that for any $t_0 \in \mathbb{R}$, a set of solutions $\{x_1, \ldots, x_k\}$ is linearly independent if and only if the set of vectors $\{x_1(t_0), \ldots, x_k(t_0)\}$ is linearly independent.

2.2. **Finding Solutions.**

To find the space of solutions of equation 2.1, we need to find n linearly independent solutions. For that, we find the eigenvalues and (generalized) eigenvectors of the matrix A as in 1.6.

An eigenvector v with an eigenvector λ gives as a solution

$$x(t) = e^{\lambda t} v$$

since

$$Ax(t) = Ae^{\lambda t} v = e^{\lambda t} Av = e^{\lambda t}(\lambda v) = \dot{x}(t).$$

If the characteristic polynomial of A has complex roots, the above argument still holds, but now the solution

$$x(t) = e^{\lambda t} v$$

is complex valued. Since A has real coefficients, both real and imaginary part of $x(t)$ are solutions of equation 2.1. If $\lambda = \alpha + i\beta$ and $v = u + iw$ where $\alpha, \beta \in \mathbb{R}$ and $v, w \in \mathbb{R}^n$, then

$$\begin{aligned} x(t) &= e^{(\alpha+i\beta)t}(u + iw) = e^{\alpha t}(\cos\beta t + i \sin\beta t)(u + iw) \\ &= e^{\alpha t}\left(\cos\beta t \cdot u - \sin\beta t \cdot w + i\left(\sin\beta t \cdot u + \cos\beta t \cdot w\right)\right) \end{aligned}$$

and we get two real solutions

$$\begin{aligned} x_1(t) &= e^{\alpha t}\left(\cos\beta t \cdot u - \sin\beta t \cdot w\right) \\ x_2(t) &= e^{\alpha t}\left(\sin\beta t \cdot u + \cos\beta t \cdot w\right). \end{aligned}$$

Since λ is a complex root, its conjugate $\bar{\lambda}$ is also a root and the above produces two linearly independent real solutions corresponding to two roots $\lambda, \bar{\lambda}$. (the two real solutions x_1, x_2 form a different basis for the complex span of the complex solutions $e^{\lambda t}v$, $e^{\bar{\lambda}t}\bar{v}$).

If the characteristic polynomial has repeated roots, there might not be n linearly independent eigenvectors of A, but we can always find n linearly independent generalized eigenvectors of A. Given a generalized eigenvector v corresponding to an eigenvalue λ, i.e. satisfying

$$(A - \lambda I)^k v = 0$$

for some $k \in \mathbb{N}$, a solution of equation 2.1 is

$$x(t) = e^{\lambda t}(v + t(A-\lambda)v + \frac{t^2(A-\lambda)^2 v}{2!} + \cdots + \frac{t^{k-1}(A-\lambda)^{k-1}v}{(k-1)!}).$$

In the simplest case where $(A - \lambda I)^2 v = 0$, this takes the form

$$x(t) = e^{\lambda t}(v + t(A-\lambda)v).$$

Note that an eigenvector is also a generalized eigenvector and this solution coincides with the one written before for eigenvectors since in that case the term $(A - \lambda)v$ vanishes.

3. PDEs

3.1. **Fourier Series.**

Theorem 18. *Let f be a function on $[-l, l]$ such that both f and f' are piecewise continuous. Define constants a_n, b_n for $n > 0$ and a_0 by*

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx.$$

We then have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right).$$

To be more precise, the series converges for all values of $x \in [-l, l]$ and equals $f(x)$ if f is continuous at x. (look on pg 488 for the behavior at points of discontinuity.)

For the solutions of the heat equation and the wave equation we will need to expand a function in terms of only sine functions or only cosine functions. Below are the corresponding statements which are consequences of the above theorem.

Theorem 19. *Let f be a function on $[0, l]$ such that both f and f' are piecewise continuous. Define constants a_n for $n \geq 0$ by*

$$a_n = \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}dx.$$

Then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}.$$

To be more precise, the series converges for all values of $x \in [0, l]$ and equals $f(x)$ if f is continuous at x.

Theorem 20. *Let f be a function on $[0, l]$ such that both f and f' are piecewise continuous. Define constants b_n for $n > 0$ by*

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx.$$

Then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}.$$

To be more precise, the series converges for all values of $x \in [0, l]$ and equals $f(x)$ if f is continuous at x.

<table>
<tr><td colspan="4">Look for $u(x,t)$ where $0 \le x \le l, 0 \le t$</td></tr>
<tr><td colspan="2">Heat equation:
$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$</td><td colspan="2">Wave equation:
$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$</td></tr>
<tr><td colspan="4">Boundary Conditions:
Dirichlet: $u(0,t) = u(l,t) = 0$ for all t
Neumann: $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(l,t) = 0$ for all t</td></tr>
<tr><td colspan="4">Solutions:</td></tr>
<tr><td colspan="2">$u_n = X_n(x)T_n(t)$
$T_n(t) = e^{\frac{-\alpha^2 n^2 \pi^2}{l^2} t}$</td><td colspan="2">$u_n = X_n(x)T_n(t)$
$u'_n = X_n(x)T'_n(t)$
$T_n(t) = \cos(\frac{cn\pi t}{l})$
$T'_n(t) = \sin(\frac{cn\pi t}{l})$</td></tr>
<tr><td>Dirichlet</td><td>Neumann</td><td>Dirichlet</td><td>Neumann</td></tr>
<tr><td>$n \ge 1,\ X_n(x) = \sin \frac{n\pi x}{l}$</td><td>$n \ge 0,\ X_n(x) = \cos \frac{n\pi x}{l}$</td><td>$n \ge 1,\ X_n(x) = \sin \frac{n\pi x}{l}$</td><td>$n \ge 0,\ X_n(x) = \cos \frac{n\pi x}{l}$</td></tr>
<tr><td colspan="2">General Solution
$u(x,t) = \sum_n a_n u_n(x,t)$</td><td colspan="2">General Solution
$u(x,t) = \sum_n a_n u_n(x,t) + b_n u'_n(x,t)$</td></tr>
<tr><td colspan="2">Initial value problem
$u(x,0) = f(x)$</td><td colspan="2">Initial value problem
$u(x,0) = f(x)$
$\frac{\partial u}{\partial t}(x,0) = g(x)$</td></tr>
<tr><td>$\sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} = f(x)$</td><td>$\sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{l} = f(x)$</td><td>$\sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} = f(x)$
$\sum_{n=1}^{\infty} b_n \frac{cn\pi}{l} \sin \frac{n\pi x}{l} = g(x)$</td><td>$\sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{l} = f(x)$
$\sum_{n=0}^{\infty} b_n \frac{cn\pi}{l} \cos \frac{n\pi x}{l} = g(x)$</td></tr>
</table>

Table 1.

3.2. **Heat/Wave Equations.**